

A TORELLI THEOREM FOR MODULI SPACES OF PRINCIPAL BUNDLES ON CURVES DEFINED OVER \mathbb{R}

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ABSTRACT. Let X be a geometrically irreducible smooth projective curve, of genus at least three, defined over the field of real numbers. Let G be a connected reductive affine algebraic group, defined over \mathbb{R} , such that G is nonabelian and has one simple factor. We prove that the isomorphism class of the moduli space of principal G -bundles on X determine uniquely the isomorphism class of X .

1. INTRODUCTION

Let X be a geometrically irreducible smooth projective curve defined over the field of real numbers, of genus g , with $g \geq 3$. Let $\mathcal{L} \in \text{Pic}^d(X)$ be a point defined over \mathbb{R} . We note that \mathcal{L} need not correspond to a line bundle over X . For example, the unique \mathbb{R} -point of Pic^1 of the anisotropic conic does not correspond to a line bundle over the anisotropic conic. Let $\mathcal{N}_X(r, \mathcal{L})$ denote the moduli space of semistable vector bundles on X of rank r and determinant \mathcal{L} , where $r \geq 2$.

We prove that the isomorphism class of the variety $\mathcal{N}_X(r, \mathcal{L})$ uniquely determines the isomorphism class of the real curve X (Theorem 2.1).

When the base field is complex numbers, this was proved in [MN] for rank two, and in [Tj], [KP, p. 229, Theorem E] for general r and d .

Let $X_{\mathbb{C}}$ be the complexification of X . Let $G_{\mathbb{C}}$ be a connected reductive affine algebraic group defined over \mathbb{C} , and let G be a real form of $G_{\mathbb{C}}$. We assume that $G_{\mathbb{C}}$ is nonabelian and it has exactly one simple factor. The antiholomorphic involution of $G_{\mathbb{C}}$ corresponding to G will be denoted by σ_G . Let $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ denote the moduli space of topologically trivial semistable principal $G_{\mathbb{C}}$ -bundles on $X_{\mathbb{C}}$. The variety $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ is the complexification of the component of the moduli space of principal G -bundles on X that contains the trivial G -bundle. The involution σ_G and the antiholomorphic involution of $X_{\mathbb{C}}$ together produce the antiholomorphic involution $\sigma_{\mathcal{M}}$ of $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$.

We prove that the isomorphism class of the real variety $(\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}), \sigma_{\mathcal{M}})$ uniquely determines the isomorphism class of X (Theorem 3.4).

The proof of Theorem 3.4 crucially uses a result of [BHo] which says that the isomorphism class of $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ uniquely determines the isomorphism class of $X_{\mathbb{C}}$.

2010 *Mathematics Subject Classification.* 14D20, 14P99, 14C34.

Key words and phrases. Curve over \mathbb{R} , principal bundle, moduli space, semistability, Torelli theorem.

2. MODULI SPACES OF VECTOR BUNDLES

Let X be a geometrically irreducible smooth projective curve defined over \mathbb{R} . Let g denote the genus of X . We will assume that $g \geq 3$. For any $d \in \mathbb{Z}$ and any integer $r \geq 2$, let $\mathcal{M}_X(r, d)$ be the moduli space of semistable vector bundles on X of rank r and degree d ; see [BHu], [BGH], [BHH], [Sc1], [Sc2], [Sc3] for moduli spaces of bundles over X . Let

$$\det : \mathcal{M}_X(r, d) \longrightarrow \text{Pic}^d(X)$$

be the morphism defined by $E \longmapsto \bigwedge^r E$. Take any \mathbb{R} -point $\mathcal{L} \in \text{Pic}^d(X)$. Define

$$\mathcal{N}_X(r, \mathcal{L}) := \det^{-1}(\mathcal{L}) \subset \mathcal{M}_X(r, d).$$

This $\mathcal{N}_X(r, \mathcal{L})$ is a geometrically irreducible normal projective variety defined over \mathbb{R} , of dimension $(r^2 - 1)(g - 1)$.

Let $X_{\mathbb{C}} := X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ be the complex projective curve obtained from X by extending the base field to \mathbb{C} . Let $\mathcal{L}_{\mathbb{C}} \in \text{Pic}^d(X_{\mathbb{C}})$ be the pull-back of \mathcal{L} to $X_{\mathbb{C}}$ by the natural morphism $\xi : X_{\mathbb{C}} \longrightarrow X$. The nontrivial element of the Galois group $\text{Gal}(\xi) = \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ produces an antiholomorphic involution

$$\sigma : X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}.$$

The conjugate vector bundle of a holomorphic vector bundle E on $X_{\mathbb{C}}$ will be denoted by \overline{E} . We recall that the underlying real vector bundle for \overline{E} is identified with that of E , while the multiplication on \overline{E} by any $c \in \mathbb{C}$ coincides with the multiplication by \overline{c} on E . The C^∞ vector bundle $\sigma^*\overline{E}$ has a natural holomorphic structure which is uniquely determined by the condition that the natural \mathbb{R} -linear identification of it with E is antiholomorphic. Note that we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\sim} & \sigma^*\overline{E} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X \end{array}$$

It is easy to see that E is semistable (respectively, stable) if and only if $\sigma^*\overline{E}$ is semistable (respectively, stable). Similarly, E is polystable if and only if $\sigma^*\overline{E}$ is polystable.

The above hypothesis that $\mathcal{L} \in \text{Pic}^d(X)$ means that the line bundle $\mathcal{L}_{\mathbb{C}}$ is holomorphically isomorphic to the line bundle $\sigma^*\overline{\mathcal{L}_{\mathbb{C}}}$.

Let $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ be the moduli space of semistable vector bundles on $X_{\mathbb{C}}$ of rank r and determinant $\mathcal{L}_{\mathbb{C}}$. The complex variety $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ coincides with the complexification $\mathcal{N}_X(r, \mathcal{L}) \times_{\mathbb{R}} \mathbb{C}$ of $\mathcal{N}_X(r, \mathcal{L})$; the resulting antiholomorphic involution

$$\sigma_{\mathcal{N}} : \mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}) \longrightarrow \mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}) \tag{2.1}$$

sends a vector bundle E on $X_{\mathbb{C}}$ to the vector bundle $\sigma^*\overline{E}$.

Theorem 2.1. *The isomorphism class of the \mathbb{R} -variety $\mathcal{N}_X(r, \mathcal{L})$ uniquely determines the isomorphism class of the real curve X .*

Proof. First note that the isomorphism class of the complex variety $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ uniquely determines the complex curve $X_{\mathbb{C}}$ [Tj], [KP, p. 229, Theorem E]. We have to prove that the antiholomorphic involution $\sigma_{\mathcal{N}}$ determines σ .

Let τ be an antiholomorphic involution of $X_{\mathbb{C}}$ such that the involution $E \mapsto \tau^* \overline{E}$ preserves $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$. The resulting antiholomorphic involution of $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ will be denoted by $\tau_{\mathcal{N}}$. The two real varieties $(\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}), \tau_{\mathcal{N}})$ and $(\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}), \sigma_{\mathcal{N}})$ are isomorphic if and only if there exists a complex algebraic automorphism f of $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ such that

$$\tau_{\mathcal{N}} = f^{-1} \sigma_{\mathcal{N}} f. \quad (2.2)$$

Assume that the two real varieties $(\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}), \tau_{\mathcal{N}})$ and $(\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}), \sigma_{\mathcal{N}})$ are isomorphic. Fix an automorphism f of $\mathcal{N}_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ satisfying (2.2).

The dual a vector bundle E will be denoted by E^{\vee} ; the dual of a line bundle ν will also be denoted by ν^{-1} .

Take any algebraic automorphism h of $\mathcal{N}_{X_{\mathbb{C}}}(r, L_{\mathbb{C}})$. It follows from [KP, p. 228, Theorem B] and [KP, p. 228, remark 0.1] that h is either of the form $E \mapsto H^* E \otimes \nu$ or $E \mapsto H^* E^{\vee} \otimes \nu_1$, where H is an automorphism of $X_{\mathbb{C}}$ uniquely determined by h while ν a line bundle on $X_{\mathbb{C}}$ with $\nu^{\otimes r} = \mathcal{O}_X$ and ν_1 a line bundle on $X_{\mathbb{C}}$ with $\nu_1^{\otimes r} = L_{\mathbb{C}}^{\otimes 2}$; it should be clarified both ν and ν_1 are independent of E . Therefore, we get a map

$$\Psi : \text{Aut}(\mathcal{N}_{X_{\mathbb{C}}}(r, L_{\mathbb{C}})) \longrightarrow \text{Aut}(X_{\mathbb{C}}), \quad h \mapsto H^{-1}. \quad (2.3)$$

It is straight-forward to check that Ψ is a homomorphism of groups.

We will denote $\Psi(f) \in \text{Aut}(X_{\mathbb{C}})$ by φ , where Ψ is defined in (2.3) and f is the automorphism in (2.2). First assume that

$$f(V) = A \otimes \varphi^* V,$$

where A is a line bundle on $X_{\mathbb{C}}$. Therefore, we have

$$f^{-1}(V) = ((\varphi^{-1})^* A^{-1}) \otimes (\varphi^{-1})^* V.$$

Hence the automorphism $\tau_{\mathcal{N}}^{-1} \circ f^{-1} \circ \sigma_{\mathcal{N}} \circ f$ of $\mathcal{N}_{X_{\mathbb{C}}}(r, L_{\mathbb{C}})$ is the morphism defined by

$$\begin{aligned} V &\mapsto \tau^* \overline{((\varphi^{-1})^* A^{-1}) \otimes (\varphi^{-1})^* ((\sigma^* \overline{A}) \otimes (\sigma^* \overline{\varphi^* V}))} \\ &= B \otimes \tau^* ((\varphi^{-1})^* \sigma^* \varphi^* V) = B \otimes (\varphi \circ \sigma \circ \varphi^{-1} \circ \tau)^* V, \end{aligned}$$

where B is a line bundle which does not depend on V . This implies that

$$\eta := \Psi(\tau_{\mathcal{N}}^{-1} \circ f^{-1} \circ \sigma_{\mathcal{N}} \circ f) = \varphi \circ \sigma \circ \varphi^{-1} \circ \tau. \quad (2.4)$$

Now from (2.2) we conclude that $\eta = \text{Id}_{X_{\mathbb{C}}}$. So from (2.4) we have

$$\tau = \varphi \circ \sigma \circ \varphi^{-1}.$$

Therefore, φ produces an isomorphism between the two curves $(X_{\mathbb{C}}, \sigma)$ and $(X_{\mathbb{C}}, \tau)$.

Next assume that

$$f(V) = A \otimes \varphi^* V^{\vee},$$

where A is a line bundle on $X_{\mathbb{C}}$. Then

$$f^{-1}(V) = ((\varphi^{-1})^* A) \otimes (\varphi^{-1})^* V^{\vee}.$$

Therefore, the automorphism $\tau_{\mathcal{N}}^{-1} \circ f^{-1} \circ \sigma_{\mathcal{N}} \circ f$ of $\mathcal{N}_{X_{\mathbb{C}}}(r, L_{\mathbb{C}})$ is the morphism defined by

$$\begin{aligned} V &\longmapsto \overline{\tau^*((\varphi^{-1})^*A) \otimes ((\varphi^{-1})^*((\sigma^*\bar{A}) \otimes (\sigma^*\varphi^*\overline{V^V}))^\vee)} \\ &= B \otimes \tau^*(\varphi^{-1})^*\sigma^*\varphi^*V = B \otimes (\varphi \circ \sigma \circ \varphi^{-1} \circ \tau)^*V, \end{aligned}$$

where B is a line bundle which does not depend on V . This implies that

$$\Psi(\tau_{\mathcal{N}}^{-1} \circ f^{-1} \circ \sigma_{\mathcal{N}} \circ f) = \varphi \circ \sigma \circ \varphi^{-1} \circ \tau.$$

Hence, as before, $\tau = \varphi \circ \sigma \circ \varphi^{-1}$. This completes the proof of the theorem. \square

3. MODULI SPACES OF PRINCIPAL BUNDLES

Let $G_{\mathbb{C}}$ be a connected nonabelian reductive group over \mathbb{C} with only one simple factor and let

$$\sigma_G : G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}$$

be an antiholomorphic automorphism of order two. We denote by G the real form of $G_{\mathbb{C}}$ corresponding to σ_G .

Let $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ denote the moduli space of topologically trivial semistable principal $G_{\mathbb{C}}$ -bundles on $X_{\mathbb{C}}$. It is an irreducible normal projective variety defined over \mathbb{C} . For any holomorphic principal $G_{\mathbb{C}}$ -bundle E on $X_{\mathbb{C}}$, let

$$\bar{E} = E(\sigma_G) = E \times^{\sigma_G} G_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$$

be the C^∞ principal $G_{\mathbb{C}}$ -bundle obtained by twisting the action of $G_{\mathbb{C}}$ using the homomorphism σ_G . So the total space of \bar{E} is identified with that of E , but the action of any $y \in G_{\mathbb{C}}$ on \bar{E} is the action of $\sigma_G(y)$ on E in terms of the identification of E with \bar{E} . The pullback $\sigma^*\bar{E}$ has a holomorphic structure uniquely determined by the condition that the above identification between the total spaces of E and $\sigma^*\bar{E}$ is anti-holomorphic; since the total spaces of \bar{E} and $\sigma^*\bar{E}$ are naturally identified, the above identification between the total spaces of E and \bar{E} produces an identification of the total spaces of E and $\sigma^*\bar{E}$. The complex projective variety $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ carries a real structure associated to the antiholomorphic involution

$$\sigma_{\mathcal{M}} : \mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}) \longrightarrow \mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}), \quad E \longmapsto \sigma^*\bar{E}.$$

Let $\mathcal{M}_X(G)$ denote the variety over \mathbb{R} defined by the above pair $(\mathcal{M}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \sigma_{\mathcal{M}})$.

A Zariski closed connected subgroup $P \subset G_{\mathbb{C}}$ is called a parabolic subgroup if $G_{\mathbb{C}}/P$ is a complete variety. A Levi subgroup of P is a maximal connected reductive subgroup of P containing a maximal torus. Any two Levi subgroups of P are conjugate by some element of P . A proper parabolic subgroup $P \subset G_{\mathbb{C}}$ is called maximal if there is no proper parabolic subgroup of $G_{\mathbb{C}}$ containing P .

Lemma 3.1. *There exists a maximal parabolic subgroup $P \subset G_{\mathbb{C}}$ and a Levi subgroup $L \subset P$, such that the two subgroups $\sigma_G(L)$ and L are conjugate by some element of $G_{\mathbb{C}}$.*

Proof. For any parabolic subgroup $P \subset G_{\mathbb{C}}$, the image $\sigma_G(P)$ is also a parabolic subgroup of $G_{\mathbb{C}}$. Since $\sigma_G(y^{-1}Py) = \sigma_G(y)^{-1}\sigma_G(P)\sigma_G(y)$, we get a self-map of the conjugacy classes of parabolic subgroups of $G_{\mathbb{C}}$ that sends the conjugacy class of any P to the conjugacy class of $\sigma_G(P)$. Therefore, the involution σ_G also acts on the Dynkin diagram D of $G_{\mathbb{C}}$ as an involution τ . Examining the Dynkin diagrams we observe that an involution of the Dynkin diagram of $G_{\mathbb{C}}$ must have a fixed point unless $G_{\mathbb{C}}$ is of type A_n for n even.

If $G_{\mathbb{C}}$ is not of type A_n , let P be a maximal parabolic subgroup corresponding to a vertex of D fixed by the above constructed involution τ . Then P and $\sigma_G(P)$ are conjugate in $G_{\mathbb{C}}$. Let $y \in G_{\mathbb{C}}$ be such that $\sigma_G(P) = y^{-1}Py$. Then for any Levi subgroup L of P ,

$$y^{-1}Ly \subset y^{-1}Py = \sigma_G(P)$$

is a Levi subgroup of $\sigma_G(P)$.

If $G_{\mathbb{C}}$ is of type A_n , then $\sigma_G(L)$ and L are conjugate for every Levi subgroup of every maximal parabolic subgroup of $G_{\mathbb{C}}$. It is enough to check this for $G_{\mathbb{C}} = \mathrm{SL}(n+1, \mathbb{C})$, in which case this is obvious. \square

Remark 3.2. We can be more precise as follows. The two subgroups $\sigma_G(L)$ and L are conjugate for every Levi subgroup of every maximal parabolic subgroup of $G_{\mathbb{C}}$ unless $G_{\mathbb{C}}$ is of type D_n (with $n \geq 4$) or E_6 .

Lemma 3.3. *Let L be any Levi subgroup of a parabolic subgroup P of $G_{\mathbb{C}}$, and let*

$$L' = [L, L]$$

be its derived subgroup. Then the homomorphism

$$\pi_1(L') \longrightarrow \pi_1(G_{\mathbb{C}}) \tag{3.1}$$

induced by the inclusion $L' \hookrightarrow G_{\mathbb{C}}$ is injective.

Proof. Consider the fibration $L' \longrightarrow G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}/L'$. Let

$$\pi_2(G_{\mathbb{C}}) \longrightarrow \pi_2(G_{\mathbb{C}}/L') \longrightarrow \pi_1(L') \longrightarrow \pi_1(G_{\mathbb{C}}) \tag{3.2}$$

be the long exact sequence of homotopy groups associated to it. From (3.2) we conclude that the homomorphism in (3.1) is injective if

$$\pi_2(G_{\mathbb{C}}/L') = 0. \tag{3.3}$$

Since $\pi_2(G_{\mathbb{C}}) = 0$ and $\pi_1(L')$ is a finite group (recall that L' is semisimple), from (3.2) it follows that $\pi_2(G_{\mathbb{C}}/L')$ is a finite group.

Now consider the fibration $P/L' \longrightarrow G_{\mathbb{C}}/L' \longrightarrow G_{\mathbb{C}}/P$. Let

$$\pi_2(P/L') \longrightarrow \pi_2(G_{\mathbb{C}}/L') \longrightarrow \pi_2(G_{\mathbb{C}}/P) \longrightarrow \pi_1(G_{\mathbb{C}}) \tag{3.4}$$

be the long exact sequence of homotopy groups associated to it. Since $G_{\mathbb{C}}/P$ is simply connected, the second homotopy group $\pi_2(G_{\mathbb{C}}/P)$ is isomorphic to $H_2(G_{\mathbb{C}}/P, \mathbb{Z})$, which is a free abelian group. Therefore, there is no nonzero homomorphism from the finite group $\pi_2(G_{\mathbb{C}}/L')$ to $\pi_2(G_{\mathbb{C}}/P)$. Hence, the homomorphism

$$\pi_2(P/L') \longrightarrow \pi_2(G_{\mathbb{C}}/L') \tag{3.5}$$

in (3.4) is surjective.

Finally, consider the long exact sequence of homotopy groups

$$\pi_2(L/L') \longrightarrow \pi_2(P/L') \longrightarrow \pi_2(P/L) \quad (3.6)$$

associated to the fibration

$$L/L' \longrightarrow P/L' \longrightarrow P/L.$$

Since P/L is diffeomorphic to the unipotent radical of P , which is contractible, we have $\pi_2(P/L) = 0$. Also, $\pi_2(L/L') = 0$ because L/L' is a Lie group. Hence from (3.6) it follows that $\pi_2(P/L') = 0$. This implies that (3.3) holds because the homomorphism in (3.5) is surjective. \square

Theorem 3.4. *The real variety $\mathcal{M}_X(G)$ uniquely determines the real curve X .*

Proof. We already know that the complex variety $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}) = \mathcal{M}_X(G) \times_{\mathbb{R}} \mathbb{C}$ determines the complex curve $X_{\mathbb{C}}$ [BHo].

Let $\text{Sing}(\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}))$ denote the singular locus of the variety $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$. Recall from [BHo] that the *strictly semi-stable locus*

$$\Delta_G \subset \mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$$

is the Zariski closure of the set of closed points $[E] \in \text{Sing}(\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}))$ with the property that every Euclidean neighborhood U of $[E]$ contains an open neighborhood $U' \ni [E]$ such that $U' \setminus (U' \cap \text{Sing}(\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})))$ is connected and simply connected. Moreover, this closed subset Δ_G is the union of irreducible components corresponding to the conjugacy classes of Levi subgroups of maximal parabolic subgroups of $G_{\mathbb{C}}$. More precisely, the decomposition of Δ_G into irreducible components is the union

$$\Delta_G = \bigcup_L M_L$$

where L ranges over conjugacy classes of Levi subgroups of maximal parabolic subgroups of $G_{\mathbb{C}}$, and M_L is the image of the morphism $\mathcal{M}_{X_{\mathbb{C}}}(L) \longrightarrow \mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ given by the inclusion of L in $G_{\mathbb{C}}$. This can be deduced from [BHo, Proposition 3.1] and Lemma 3.3. Indeed, every closed point in Δ_G is defined by a principal $G_{\mathbb{C}}$ -bundle E admitting a reduction of structure group E_L to a Levi subgroup L of a maximal parabolic subgroup of $G_{\mathbb{C}}$. Moreover, this L -bundle E_L is semistable and its topological type $\delta \in \pi_1(L)$ is torsion, which means that δ belongs to $\pi_1([L, L])$; this is because $\pi_1(L/[L, L])$ is free abelian. Now, since E is topologically trivial, δ must be trivial (follows from Lemma 3.3), i.e., $[E] \in \mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ belongs to M_L .

Moreover, M_L is never empty, since there always exist semi-stable principal L -bundles which are topologically trivial, for example the trivial holomorphic principal L -bundle. The fact that the subvarieties M_{L_1} and M_{L_2} of $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ are distinct when L_1 and L_2 are not conjugate by some element of $G_{\mathbb{C}}$ is contained in the last part of the proof of [BHo, Proposition 3.1].

The antiholomorphic involution $\sigma_{\mathcal{M}}$ maps the strictly semi-stable locus Δ_G into itself, permuting its irreducible components. It follows from Lemma 3.1 that there exists at least one component M_L which is fixed by $\sigma_{\mathcal{M}}$. The restriction of $\sigma_{\mathcal{M}}$ to this component M_L has at least one fixed point, namely the closed point corresponding to the trivial bundle.

We now proceed as in the proof of [BHo, Theorem 4.1] to recover the involution σ defining the real curve X .

First, one can assume $G_{\mathbb{C}}$ to be semi-simple. To see this, let $Z_{G_{\mathbb{C}}}^0$ be the connected component of the center of $G_{\mathbb{C}}$ containing the identity element. Let us denote by G' the quotient $G_{\mathbb{C}}/Z_{G_{\mathbb{C}}}^0$ of $G_{\mathbb{C}}$. Note that σ_G preserves $Z_{G_{\mathbb{C}}}^0$, so it produces a real structure on the quotient G' . The canonical line bundle of $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ (respectively, $\mathcal{M}_{X_{\mathbb{C}}}(G')$) will be denoted by $\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})}$ (respectively, $\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G')}$). We note that $\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G')}$ pulls back to $\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})}$ under the morphism $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}) \rightarrow \mathcal{M}_{X_{\mathbb{C}}}(G')$ given by the quotient map $G_{\mathbb{C}} \rightarrow G'$. There exists an integer m such that the pluri-anti-canonical system $|-m\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})}|$ factors into the natural map $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}) \rightarrow \mathcal{M}_{X_{\mathbb{C}}}(G')$ followed by the embedding

$$\mathcal{M}_{X_{\mathbb{C}}}(G') \hookrightarrow |-m\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})}|^* = |-m\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G')}|^*.$$

Since the dualizing sheaves are defined over the reals, we have real structures on $|-m\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})}|^*$ and $|-m\omega_{\mathcal{M}_{X_{\mathbb{C}}}(G')}|^*$. All the maps above are defined over \mathbb{R} . Therefore, it is enough to prove the theorem for G' .

So let us assume that $G_{\mathbb{C}}$ is semi-simple. We have seen above that $\Delta_G \subset \mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ contains at least one irreducible component fixed by $\sigma_{\mathcal{M}}$, which is equal to the variety M_L associated to a Levi subgroup L of a maximal parabolic subgroup of $G_{\mathbb{C}}$. Let

$$\alpha : \widetilde{M}_L \rightarrow M_L$$

be the normalization of M_L , and let σ_L be the antiholomorphic involution of M_L . Since normalization commutes with the base change of field of definition, the variety \widetilde{M}_L is also defined over \mathbb{R} , and the morphism α is also defined over \mathbb{R} . Hence the antiholomorphic involution σ_L of M_L lifts to \widetilde{M}_L . Moreover, \widetilde{M}_L is isomorphic to the quotient $\mathcal{M}_{X_{\mathbb{C}}}(L)/\Gamma_L$, where Γ_L is the image of $N_{G_{\mathbb{C}}}(L)$ in $\text{Out}(L)$, which is either trivial or $\mathbb{Z}/2\mathbb{Z}$ (see [BHo]), and this quotient map is compatible with the real structures on $\mathcal{M}_{X_{\mathbb{C}}}(L)$ and \widetilde{M}_L .

Let Z_L^0 be the connected component of the center of L containing the identity element. Let us denote by L' the quotient L/Z_L^0 . Then the above group Γ_L also acts on $\mathcal{M}_{X_{\mathbb{C}}}(L')$, and the morphism (defined over the real numbers)

$$\theta : \widetilde{M}_L \simeq \mathcal{M}_{X_{\mathbb{C}}}(L)/\Gamma_L \rightarrow \mathcal{M}_{X_{\mathbb{C}}}(L')/\Gamma_L \quad (3.7)$$

can be recovered from the second tensor power of the canonical line bundle on the smooth locus of \widetilde{M}_L . Indeed, this second tensor power extends to a line bundle on the whole variety, and a sufficiently negative power of it gives the morphism θ (see [BHo]).

Let $\beta : \mathcal{M}_{X_{\mathbb{C}}}(L') \rightarrow \mathcal{M}_{X_{\mathbb{C}}}(L')/\Gamma_L$ be the quotient map. Consider θ in (3.7). For any point $y \in \mathcal{M}_{X_{\mathbb{C}}}(L')/\Gamma_L$, the fiber $\theta^{-1}(y)$ is $J_{X_{\mathbb{C}}}$ (respectively, $J_{X_{\mathbb{C}}}(\mathbb{Z}/2\mathbb{Z})$) if $\#\beta^{-1}(y) = 2$ (respectively, $\#\beta^{-1}(y) = 1$).

Now take any smooth point

$$y \in \mathcal{M}_{X_{\mathbb{C}}}(L')/\Gamma_L$$

fixed by the antiholomorphic involution. As noted above, the fiber $\theta^{-1}(y)$ is isomorphic to either $J_{X_{\mathbb{C}}}$ or the singular Kummer variety $J_{X_{\mathbb{C}}} / (\mathbb{Z}/2\mathbb{Z})$. The real structure on $\theta^{-1}(y)$ induced by that of \widetilde{M}_L comes from the real structure on the Jacobian associated to the curve. So in both cases we recover the Jacobian variety together with its natural real structure: when $\theta^{-1}(y)$ is isomorphic to $J_{X_{\mathbb{C}}} / (\mathbb{Z}/2\mathbb{Z})$, then J_X is obtained from the two-sheeted cover of the smooth locus of the Kummer variety defined by the unique maximal torsion-free subgroup in its fundamental group. The antiholomorphic involution can be lifted to this cover, and this lift extends to $J_{X_{\mathbb{C}}}$ because its construction is over \mathbb{R} .

Finally, the class of the canonical principal polarization on $J_{X_{\mathbb{C}}}$ is determined as in [BHo]. Now the theorem follows from the real analog of Torelli theorem [GH, Theorem 9.4]. \square

ACKNOWLEDGEMENTS

We thank the referee for helpful comments. The first author acknowledges support of a J. C. Bose Fellowship.

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